Solutions to Problem Set 2

1. (a) The mass is the integral of the density over the volume. In cylindrical coordinates this is

$$M = \int_0^\infty dRR \int_{-\infty}^\infty dz \int_0^{2\pi} d\phi \frac{\Sigma_0}{h_Z} e^{-|z|/h_Z} e^{-R/h_R}.$$

The angular integral gives a factor of 2π. The integral over z is

$$\int_{infty}^\infty dz e^{-|z|/h_Z} = 2 \int_0^\infty e^{-z/h_Z}$$

since it is symmetric over $\pm z$, so we simply do the positive part, and multiply by two. The positive part is equal to h_Z, so we now have

$$M = 4\pi\Sigma_0 \int_0^\infty dRRe^{-R/h_R}.$$

The R integral has dimensions of lengths squared, the only length is h_R, so it goes as h_R^2. We've done the resulting dimensionless integral in class: it is simply one, so

$$M = 4\pi\Sigma_0 h_R^2.$$

(b) The mass contained within a spherical radius r is simplified considerably if the disk is very thin. Then, the mass in this volume is simply the integral of the surface density $2\Sigma_0 e^{-R/h_R}$ over the disk out to radius r:

$$M(r) = \int_0^r dRR \int_0^{2\pi} d\phi 2\Sigma_0 e^{-R/h_R}.$$

Define $x \equiv R/h_R$, so that $dR = h_R dx$; then

$$M(r) = 4\pi\Sigma_0 h_R^2 \int_0^{r/h_R} dx x e^{-x}.$$

Integrate by parts with $u = x$ and $dv = dxe^{-x}$. Then,

$$\int_0^{r/h_R} dx x e^{-x} = -xe^{-x}\Big|_0^{r/h_R} + \int_0^{r/h_R} dx e^{-x}.$$

In the first term, only the upper surface term contributes, and the second integral is $-e^{-x}$, so

$$\int_0^{r/h_R} dx x e^{-x} = -\frac{r}{h_R} e^{-r/h_R} - e^{-x}\Big|_0^{r/h_R}.$$

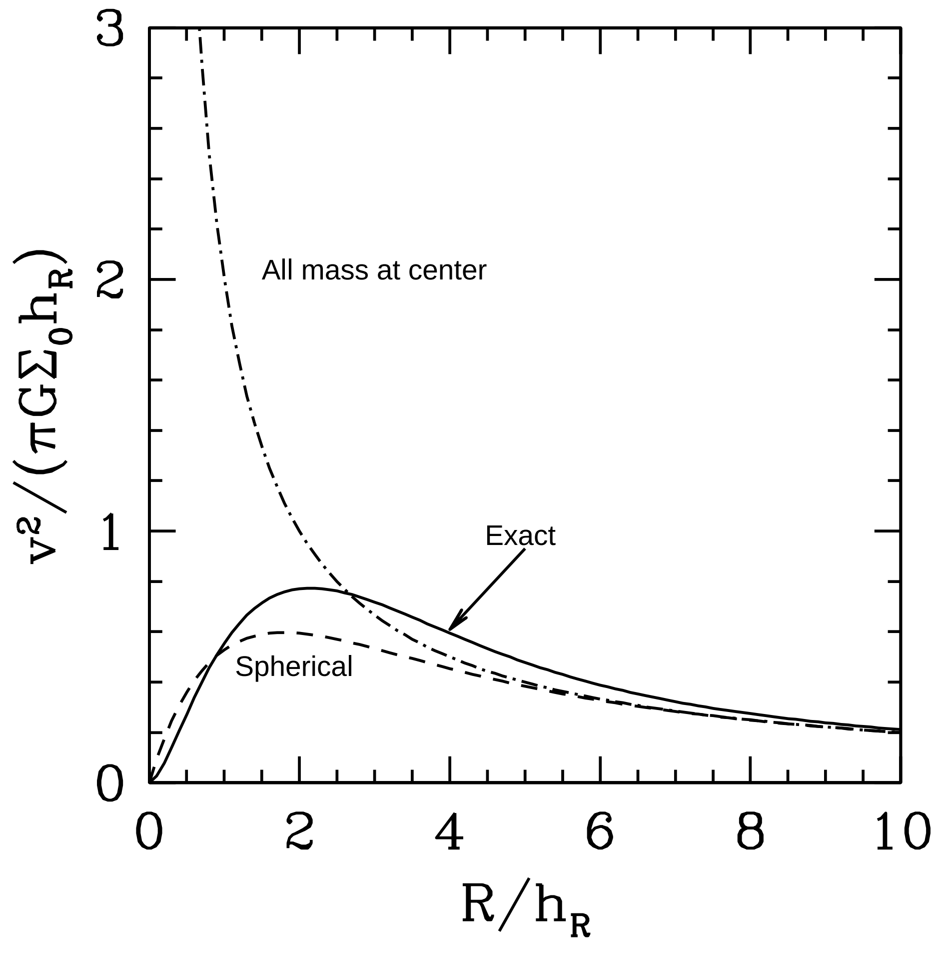
Evaluating e^{-x} at the r/h_R and 0 and plugging back in leads to

$$M(r) = 4\pi\Sigma_0 h_R^2 \left[-\frac{r}{h_R} e^{-r/h_R} - e^{-r/h_R} + 1 \right].$$

(c) The velocity due to a point mass is simply $GM(r)/r$, so

$$v^2(r) = 4\pi G\Sigma_0 \frac{h_R^2}{r} \left[-\frac{r}{h_R} e^{-r/h_R} - e^{-r/h_R} + 1 \right].$$

2. The three curves in the figure correspond to the exact Bessel function expression; the approximation that the distribution is spherical (from Problem 1) and the even more absurd approximation that all the mass is concentrated at a point in the center of the galaxy.



3. (b) Let's first write down the theoretical prediction for the velocity squared as a function of radius. The contributions from the disk and from the dark matter halo sum in quadrature, so

$$(v^{\rm theory}[R,\Sigma_0,\rho_0])^2 = 2\pi G\Sigma_0 \frac{R^2}{h_R}\left[I_0(y)K_0(y) - I_1(y)K_1(y)\right] + 4\pi G\rho_0 r_0^2\left[1 - \frac{r_0}{R}\arctan(R/r_0)\right]$$

where $y = R/2h_R$ and the second term from dark matter was derived in class. In this case both $h_R(= 2.13$ kpc) and $r_0(= 5$ kpc) are fixed, so when comparing with the data we need only minimize the χ^2 with respect to the two parameters Σ_0 and ρ_0.

To proceed, let's write the above in a more compact form:

$$(v^{\rm theory}[R])^2 = A(R)\Sigma_0 + B(R)\rho_0$$

where the functions A and B are

$$A(R) \equiv 2\pi G\frac{R^2}{h_R}\left[I_0(y)K_0(y) - I_1(y)K_1(y)\right]$$

and

$$B(R) \equiv 4\pi G r_0^2\left[1 - \frac{r_0}{R}\arctan(R/r_0)\right].$$

Again, the key point for what follows is that if you give me R I'll tell you A and B. We can now write the χ^2 as

$$\chi^2(\rho_0,\Sigma_0) = \sum_{i=1}^{N}\left(v_i^2 - A(R_i)\Sigma_0 - B(R_i)\rho_0\right)^2,$$

where i now labels all the $N = 395$ data points; e.g. $R_1 = 0$ and $v_1 = 47.9$ km/sec. We want to minimize the χ^2 with respect to the variables Σ_0 and ρ_0, so we differentiate it first with respect to Σ_0 and then set to zero:

$$\frac{\partial\chi^2}{\partial\Sigma_0} = -2\sum_i\left(v_i^2 - A(R_i)\Sigma_0 - B(R_i)\rho_0\right)A(R_i) = 0.$$

Then do the ssame thing with respect to ρ_0:

$$\frac{\partial\chi^2}{\partial\Sigma_0} = -2\sum_i\left(v_i^2 - A(R_i)\Sigma_0 - B(R_i)\rho_0\right)B(R_i) = 0.$$

We now have two equation for two unknowns. Let's solve them to find the extremum of the χ^2. The first equation can be rearranged to give

$$\rho_0 = \frac{\sum_i\left(v_i^2 A(R_i) - A(R_i)^2\Sigma_0\right)}{\sum_i B(R_i)A(R_i)} \tag{1}$$

There are going to be lots of sums over $A(R_i)$ and $B(R_i)$. To simplify the notation, let's define

$$\langle AA\rangle \equiv \sum_i A(R_i)A(R_i)$$

and similarly for $\langle AB\rangle$ and $\langle BB\rangle$. Then,

$$\rho_0 = \frac{\langle vvA\rangle - \langle AA\rangle\Sigma_0}{\langle AB\rangle}.$$

Simimlarly the second equation can be rewritten as

$$\Sigma_0 = \frac{\langle vvB\rangle - \langle BB\rangle\rho_0}{\langle AB\rangle}.$$

Into this equation, plug in our expression for ρ_0 to get

$$\Sigma_0 = \frac{\langle vvB\rangle - \langle BB\rangle\left[\frac{\langle vvA\rangle - \langle AA\rangle\Sigma_0}{\langle AB\rangle}\right]}{\langle AB\rangle}$$

Moving the Σ_0 term on the right over to the left leads to

$$\Sigma_0\left[1 - \langle BB\rangle\langle AA\rangle\langle AB\rangle^2\right] = \frac{\langle vvB\rangle\langle AB\rangle - \langle BB\rangle\langle vvA\rangle}{\langle AB\rangle^2}$$

and then multiplying both sides by $\langle AB\rangle^2$ and dividing by the ressulting term in square brackets on the left leads to

$$\Sigma_0 = \frac{\langle vvB\rangle\langle AB\rangle - \langle BB\rangle\langle vvA\rangle}{\langle AB\rangle^2 - \langle BB\rangle\langle AA\rangle}$$

an explicit expression for the surface density of the disk in terms of sums over the data and over A and B.

When I do the sums, I get:

$$\langle vvA\rangle \equiv \sum_i v_i v_i A(R_i) = 166\left(\text{km sec}^{-1}\right)^4 M_\odot^{-1}\text{kpc}^2$$

and

$$\langle vvB\rangle \equiv \sum_i v_i v_i B(R_i) = 3634\left(\text{km sec}^{-1}\right)^4 M_\odot^{-1}\text{kpc}^3$$

for the sums over the velocities. The other sums are

$$\langle AA \rangle = 33.5 \times 10^{-8} \left(\text{km sec}^{-1}\right)^4 M_\odot^{-2} \text{kpc}^4$$

and

$$\langle BB \rangle = 1.42 \times 10^{-4} \left(\text{km sec}^{-1}\right)^4 M_\odot^{-2} \text{kpc}^6$$

and the cross-term

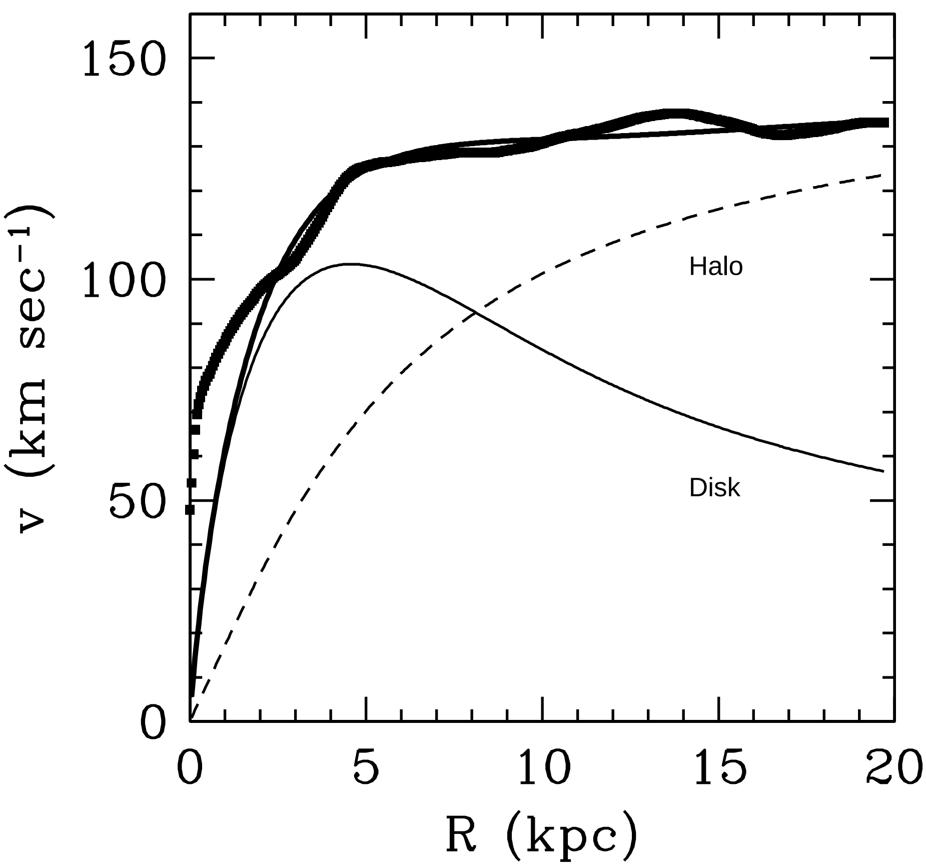
$$\langle AB \rangle = 5 \times 10^{-6} \left(\text{km sec}^{-1}\right)^4 M_\odot^{-2} \text{kpc}^5.$$

So plugging in, I get

$$\Sigma_0 = 2.4 \times 10^8 M_\odot \text{kpc}^{-2}$$

Plugging back into Eq. 1 gives

$$\rho_0 = 1.7 \times 10^7 M_\odot \text{kpc}^{-3}$$



Problem 24.1 Let's do this the GR way. The geodesic equation is

$$\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}.$$

In cartesian coordinates the Christoffel symbol is zero, so both x and y satisfy $d^2x^i/dt^2 = 0$. Now let's consider polar coordinates, in which $x^1 = R, x^2 = \theta$. It is straightforward to show that

$$\Gamma^1{}_{22} = -R \qquad ; \qquad \Gamma^2{}_{21} = \Gamma^2{}_{12} = \frac{1}{R}$$

and all other components vanish. Then,

$$\frac{dx^1}{dt^2} = \ddot{R} = -\Gamma^1{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}.$$

The only non-zero component of $\Gamma^1{}_{jk}$ is with $j = k = 2$, so

$$\ddot{R} = -\Gamma^1{}_{22}\left(\dot{\theta}\right)^2 = R\dot{\theta}^2.$$

This is indeed the equation for R. Th equation for θ is

$$\frac{d^2x^2}{dt^2} = \ddot{\theta} = -\Gamma^2{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}.$$

One of the indices must be equal to 1 and the other to 2 for $\Gamma^2{}_{jk}$ to be non-zero. Both terms contribute equally leaving

$$\ddot{\theta} = -\frac{2}{R}\dot{R}\dot{\theta}.$$

This can be rewritten as

$$\ddot{\theta} + \frac{2}{R}\dot{R}\dot{\theta} = \frac{1}{R}\frac{d}{dt}\left(R^2\dot{\theta}\right) = 0$$

the correct equation for conservation of angular momentum.